Theory and notation

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1 Abstract

This vignette sets out the formulas of the package: the user cost of a monetary asset, the expenditure shares, the Divisia index in the discrete form of Tornqvist and Theil, the Fisher ideal index, the simple sum and the currency equivalent index, the dual user-cost price index and the Divisia second moments, with the notation used throughout.

2 Notation

symbol	meaning	argument
n	number of components	x
T	number of dates	.data
i, j	components	
t	dates	date
x_{it}	quantity of component i at date t	x

symbol	meaning	argument
r_{it}	rate of return of component i at date t	r
R_t	benchmark rate at date t	benchmark
π_{it}	user cost of component i: (R_t - r_it) / (1 + R_t)	user_cost
s_{it}	share of component i in the expenditure on monetary services	
$\bar{s}_{it}$	average of the shares at t - 1 and t	
$\mathbf{x}_t$	quantities of the components at date t	x
$\boldsymbol{\pi}_t$	user costs at date t	user_cost
$\mathbf{s}_t$	expenditure shares at date t	
$\mathbf{X}$	T x n matrix of quantities (dates in rows)	x
Q_t	index of the aggregate at date t (base at the first date)	base
V_t	aggregate in the units of x (equal to the simple sum at the first date)	
E_t	expenditure on monetary services: sum_i pi_it x_it	
P_t	dual user-cost price index: E_t / V_t	
κ_t	Divisia mean of the growth of the components	
K_t	Divisia variance of the growth of the components	
Δ	first difference over dates	

3 User costs

A monetary asset gives two things: a yield and monetary services (liquidity). Barnett (1978) prices the services by the yield given up. With the benchmark rate R_t, the return on an asset that gives no monetary services, and the own rate r_{it} of component i, the user cost of one unit of it held over period t is

$$\pi_{it} = \frac{R_t - r_{it}}{1 + R_t},$$

the discounted interest forgone. The expenditure on monetary services is $E_t = \sum_i \pi_{it} x_{it}$ and the share of component i

$$s_{it} = \frac{\pi_{it} x_{it}}{E_t}.$$

The factor $1/(1+R_t)$ is the same for every component and cancels in the shares. Rates may be in percent (`percent = TRUE`) or decimal.

The user costs and shares at the first date of the simulated data:

```
data(money_sim)
uc <-
  dmai_user_cost(
```

```
      .data      = money_sim
    , date       = "Date"
    , x          = NULL
    , r          = NULL
    , benchmark = "rb"
    , percent    = TRUE
    )
uc$components[Date == min(Date)]
```

```
         Date component quantity rate  user_cost      share
1: 2005-01-01        x1  1235.89  0.0 0.08675799 0.32336335
2: 2005-01-01        x2   902.17  0.0 0.08675799 0.23604748
3: 2005-01-01        x3  1528.41  3.5 0.05479452 0.25256810
4: 2005-01-01        x4   391.09  5.5 0.03652968 0.04308480
5: 2005-01-01        x5   363.04  6.0 0.03196347 0.03499531
6: 2005-01-01        x6   453.61  6.5 0.02739726 0.03747928
7: 2005-01-01        x7   294.87  7.2 0.02100457 0.01867866
8: 2005-01-01        x8   244.10  1.5 0.07305936 0.05378301
```

The benchmark rate. R_t should exceed the rate of every component. With `benchmark = NULL` the package takes $R_t = \max_i r_{it}$, as version 0.5.0 did; the component with that rate then has a user cost of zero. With a given benchmark the package uses $\max(R_t, \max_i r_{it})$, the upper envelope.

4 Indexes

All indexes are chained: each date is linked to the previous one and the links are multiplied, with Q_1 equal to `base`. With $\bar{s}_{it} = (s_{it} + s_{i,t-1})/2$:

Divisia (`"Barnett"`). Barnett (1980) aggregates by the Divisia index, in discrete time the index of Tornqvist (1936) and Theil (1967):

$$\log \frac{Q_t}{Q_{t-1}} = \sum_{i=1}^{n} \bar{s}_{it} \left(\log x_{it} - \log x_{i,t-1} \right).$$

Divisia with simple growth (`"Hancock"`). The same shares with simple growth rates, as in version 0.5.0:

$$\frac{Q_t}{Q_{t-1}} - 1 = \sum_{i=1}^{n} \bar{s}_{it} \left(\frac{x_{it}}{x_{i,t-1}} - 1 \right).$$

Fisher ideal (`"Fisher"`). The geometric mean of the Laspeyres and Paasche quantity indexes with the user costs as prices (Fisher 1922):

$$\frac{Q_t}{Q_{t-1}} = \left(\frac{\boldsymbol{\pi}_{t-1}^{\top} \mathbf{x}_t}{\boldsymbol{\pi}_{t-1}^{\top} \mathbf{x}_{t-1}} \cdot \frac{\boldsymbol{\pi}_t^{\top} \mathbf{x}_t}{\boldsymbol{\pi}_t^{\top} \mathbf{x}_{t-1}} \right)^{1/2}.$$

The Divisia (Tornqvist) and Fisher indexes are both superlative: exact for flexible aggregator functions (Diewert 1976), and they agree closely in practice.

Simple sum (`"SimpleSum"`). $\sum_i x_{it}$, the usual official aggregate: perfect substitutes, equal weights.

Currency equivalent (`"CE"`). Rotemberg, Driscoll and Poterba (1995):

$$CE_t = \sum_{i=1}^{n} \frac{R_t - r_{it}}{R_t}\, x_{it},$$

the stock of currency that would give the same services.

Growth. `DMAI` is $100 \log(Q_t/Q_{t-1})$ for every method except `"Hancock"`, for which it is $100(Q_t/Q_{t-1} - 1)$. With `logbase` the logarithm is taken to that base; version 0.5.0 used base 10 by default, which gives $1/\log 10 \approx 0.434$ times the growth rate.

Level in money units. $V_t = V_1 Q_t/Q_1$ with $V_1 = \sum_i x_{i1}$ (column `value`).

5 The dual price index

By factor reversal the user-cost price of the aggregate is

$$P_t = \frac{E_t}{V_t},$$

so that price times quantity is the expenditure (component `price`). For the Fisher index it equals the Fisher price index of the user costs; for the Divisia index it is the implicit price index.

6 Second moments

Theil (1967) reads the shares as probabilities: the Divisia growth is then the mean of the growth rates of the components, and their variance

$$K_t = \sum_{i=1}^{n} \bar{s}_{it} \left(\Delta \log x_{it} - \kappa_t\right)^2, \qquad \kappa_t = \sum_{i=1}^{n} \bar{s}_{it}\, \Delta \log x_{it},$$

measures how differently the components grow. Barnett and Serletis (1990) use it as a diagnostic: when it is large the weights matter and an aggregate with the wrong weights (the simple sum) misleads. `dmai_moments()` gives $100\kappa_t$, $10^4 K_t$ and $100\sqrt{K_t}$, and the same for the user costs.

7 References

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